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Decomposition of Balanced Matrices.
Part VI: Even Wheels

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1 Introduction

In this part, we prove that if a balanced bipartite graph G contains an even wheel as an induced subgraph, then it has an extended star cutset. The proof is divided into two parts, treated in Sections 2 and 3 respectively. In Section 2, we give properties of the strongly adjacent nodes to even wheels. In particular, for an even wheel (W, v) with the smallest number of spokes and, subject to this, the smallest number of nodes, we prove that at least two nodes of W, say w_1 and w_2, are adjacent to all nodes with more than two neighbors in W. In Section 3, we prove that, for the above choice of W and an appropriate choice of v, the nodes with more than two neighbors in W together with the nodes of $N(v)$ form an extended star cutset of G.

2 Strongly Adjacent Nodes to an Even Wheel

Let (W, v) be an even wheel of a balanced bipartite graph G. Throughout this part, we assume that $v \in V^r$. We denote by $N_W(v)$ the set of nodes of W adjacent to v. We recaal from Part I that a subpath of W having two nodes of $N_W(v)$ as endnodes and only nodes of $V(W)\backslash N_W(v)$ as intermediate nodes is called a *sector* of (W, v). Two sectors are *adjacent* if they have a common endnode and two nodes of $N_W(v)$ are *consecutive* if they are the endnodes of some sector. Finally, we paint the nodes of $V(W)\backslash N_W(v)$ with two colors, say blue and green, in such a way that two nodes of $V(W)\backslash N_W(v)$ have the same color if they are in the same sector, and have distinct colors if they are in adjacent sectors. The nodes of $N_W(v)$ are left unpainted.

The goal of this section is to prove the following two theorems about the strongly adjacent nodes to W.

Theorem 2.1 *Let $u \in V^c\backslash N(v)$ be a node with neighbors in at least two distinct sectors of W. Then u has exactly two neighbors in W, say u_j and u_k. Furthermore, the nodes u_j, u_k belong to sectors of the same color and at least one of u_j, u_k is not adjacent to unpainted nodes.*

Define the set of nodes:

$A(W,v) = \{u \in V^r|$ No sector of (W,v) entirely contains the set $N_W(u)$ and $|N_W(v) \cap N_W(u)| \geq 2\}$.

Theorem 2.2 *If a balanced bipartite graph contains an even wheel, then it contains an even wheel* (W,v) *such that*

$$\left| \bigcap_{u \in A(W,v)} N_W(u) \right| \geq 2.$$

Proof of Theorem 2.1: Assume u has neighbors in at least three different sectors, say S_i, S_j, S_k. If none of these sectors is adjacent to the other two, then there exist three unpainted nodes v_i, v_j, v_k, such that $v_i \in V(S_i) \setminus (V(S_j) \cup V(S_k))$, $v_j \in V(S_j) \setminus (V(S_i) \cup V(S_k))$, $v_k \in V(S_k) \setminus (V(S_i) \cup V(S_j))$. This implies the existence of a $3PC(u,v)$, where each of the nodes v_i, v_j, v_k belongs to a distinct path of the 3-path configuration, see Figure 1. Now assume some sector is adjacent to the other two, say S_j is adjacent to both S_i and S_k. Then, with the notation of Figure 2, there is a $3PC(u,v)$ unless node u has a neighbor u_i in S_i adjacent to v_i and a neighbor u_k in S_k adjacent to v_k. When this is the case, the nodes u, u_i, v_i, v, v_k, u_k induce a 6-hole.

So u has neighbors in at most two different sectors of the wheel, say S_j and S_k. If these two sectors are adjacent, then denote by v_j and v_k the endnodes such that $v_j \in V(S_j) \setminus V(S_k)$ and $v_k \in V(S_k) \setminus V(S_j)$ respectively. Among the nodes of $N(u) \cap V(S_j)$, let u_j be the one such that the $u_j v_j$-subpath of S_j is shortest. Similarly $u_k \in N(u) \cap V(S_k)$ has the shortest $u_k v_k$-subpath in S_k. Let P' be the $u_j u_k$-subpath of W containing the common endnode of S_j and S_k. Now, consider the hole W' obtained from W by replacing P' by the path u_j, u, u_k. The wheel (W', v) is an odd wheel. So the sectors S_j and S_k are not adjacent.

If u has three neighbors or more on W, say two or more in S_j and at least one in S_k, then denote by v_j and v_{j-1} the endnodes of S_j and by v_k one of

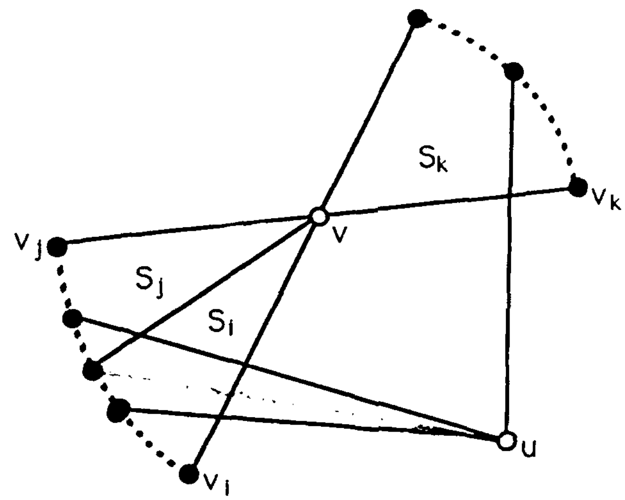

Figure 1:

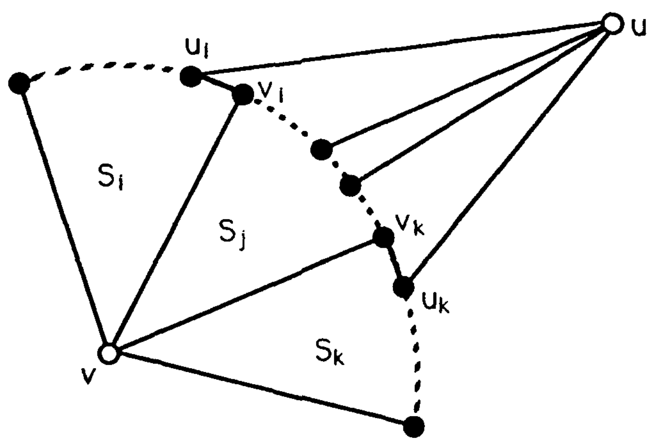

Figure 2:

the endnodes of S_k. There exists a $3PC(u,v)$ where each of the nodes v_j, v_{j-1}, and v_k belongs to a different path. Therefore u has only two neighbors in W, say $u_j \in V(S_j)$ and $u_k \in V(S_k)$. Let C_1 and C_2 be the holes formed by the node u and the two $u_j u_k$-subpaths of W, respectively. In order for both (C_1, v) and (C_2, v) to be even wheels, the sectors S_j and S_k must be of the same color. Finally, assume that u_j is adjacent to an endnode of S_j, say v_j, and that u_k is adjacent to an endnode of S_k, say v_k. Then the nodes u, v, u_j, v_j, u_k, v_k induce a 6-hole. □

Before proving Theorem 2.2, we need the following results about the structure of a strongly adjacent node $u \in V^r$ to W, relative to a chosen center v of the wheel.

Type 1: There exists a sector of (W, v) containing all the nodes of $N_W(u)$.

Type 2: Node u is not of Type 1 and all its neighbors in W are unpainted. Note that, in particular, the center v of the wheel is of Type 2.

Type 3: Node u is not of Types 1 or 2 and all its painted neighbors in W have the same color.

Type 4: Node u has painted neighbors of both colors.

We first study the structure of the Type 4 nodes.

Lemma 2.3 *Let u be a Type 4 strongly adjacent node to an even wheel (W, v). Let s and t be a green and a blue neighbor of u, respectively. Each of the st-subpaths of W contains at least one unpainted neighbor of u. Hence $u \in A(W, v)$.*

Proof: Assume that one of the two st-subpaths of W contains no unpainted neighbor of u. Let Q be this subpath. Let P be a $s't'$-subpath of Q such that s' is a green neighbor of u, t' is a blue neighbor of u, and P contains no other painted neighbor of u. P contains an odd number of unpainted nodes, none of which are adjacent to u. If this number is three or more, then v is the center of an odd wheel with hole induced by the nodes of P and u.

So P contains exactly one neighbor of v, say x. Consider the cycle C with unique chord vx induced by v and the two sectors of W having x as an endnode. Node u is a strongly adjacent node relative to C and therefore must be of one of the three types of Theorem 3.4(I). It is not of Type 1[3.4(I)] since s' and t' are in different sectors. It is not of Type 2[3.4(I)] either since u is not adjacent to v or x. Since s' and t' are painted, they are not adjacent to v and since $s', t', x \in V^c$, the nodes s' and t' are not adjacent to x either. So the node u is not of Type 3[3.4(I)] relative to C. This contradicts the balancedness assumption. Therefore Q must contain an unpainted node adjacent to u. □

Lemma 2.4 *If a node of V^r, strongly adjacent to W, has a unique neighbor w in a sector, then node w is unpainted.*

Proof: Assume that node u has a unique neighbor w in sector S_i and that w is painted. Let v_i and v_{i-1} be the endnodes of S_i. Since node u is strongly adjacent to W, it has at least one neighbor in the path induced by $V(W) \setminus V(S_i)$. Choose u^* among the nodes of $N_W(u) \setminus V(S_i)$ and choose v^* among the nodes of $N_W(v) \setminus V(S_i)$ in such a way that the u^*v^*-subpath of W not containing S_i is shortest. Note that $u^* \in V^c$, hence u^* cannot be adjacent to v_i or v_{i-1}. This implies a $3PC(w, v)$, where each of the nodes v_i, v_{i-1} and v^* belongs to a different path. □

We now consider a wheel (W, v) of G with the following property.

Definition 2.5 *A hole W is* small *if there exist no nodes $x, y, z \in V^r$, strongly adjacent to W, such that the graph induced by $V(W) \cup \{x, y, z\}$ contains a wheel (W', u) where W' is a shorter hole than W and u is one of the nodes x, y or z. A wheel (W, v) is* small *if W is small.*

For example, the shortest hole W for which there exists a node $v \in V^r \setminus V(W)$ with more than two neighbors in W, is a small hole. The following remarks are an immediate consequence of Definition 2.5.

Remark 2.6 *Every strongly adjacent node to W in V^τ which does not belong to $N(v)$ has at most two neighbors in each sector of (W, v).*

Remark 2.7 *Every Type 1 node u has exactly two neighbors in W, say u' and u'', and there is a painted node in W which is adjacent to both u' and u''.*

Remark 2.8 *For every Type 2 node u, $N_W(u) = N_W(v)$.*

Remark 2.9 *Every Type 3 node u has exactly two neighbors either in each green sector or in each blue sector. Hence, $|N_W(u)| = |N_W(v)|$.*

Lemma 2.10 *Let (W, v) be a small even wheel. Every Type 4 node u satisfies $|N_W(u)| = |N_W(v)|$.*

Proof: By Lemma 2.3, we have that $|N_W(u) \cap N_W(v)| \geq 2$. Let P be a subpath of W with endnodes in $N_W(u) \cap N_W(v)$, say x and y but no intermediate node in $N_W(u) \cap N_W(v)$. The nodes x and y are said to be consecutive nodes of $N_W(u) \cap N_W(v)$ in W. Now Lemma 2.3 implies that $V(P) \cap N(u)$ does not contain nodes of distinct colors. Assume w.l.o.g. that $V(P) \cap N(u)$ contains no green node. Then Definition 2.5 implies that u has exactly two neighbors in every blue sector of P. This shows that $|N_W(u) \cap V(P)| = |N_W(v) \cap V(P)|$. By repeating the argument between any pair of consecutive nodes of $N_W(u) \cap N_W(v)$ in W, we get the equality claimed in the lemma. □

Lemma 2.11 *Let (W, v) be a small even wheel and let u be a Type 4 node having painted neighbors in two adjacent sectors, say S_i, S_{i+1}. Then every Type 2, 3 or 4 node is adjacent to the common endnode of S_i, S_{i+1}.*

Proof: We use the notation of Figure 3. Node v_i belongs to $N(u)$, as a consequence of Lemma 2.3. Assume by contradiction that there exists a node w, of Type 2, 3 or 4, which is not adjacent to v_i.

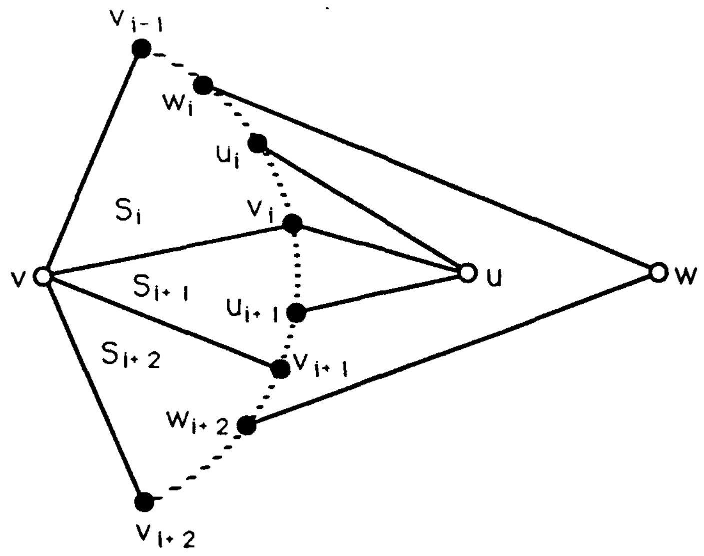

Figure 3:

By Definition 2.5, node w has a painted neighbor in S_i or S_{i+1}. If w has a painted neighbor in both S_i and S_{i+1}, then Lemma 2.3 implies that w is adjacent to v_i. Therefore we assume w.l.o.g. that w has a painted neighbor in S_i but no painted neighbor in S_{i+1}. Definition 2.5 implies that w has a neighbor in S_{i+2}. Let w_i be the painted neighbor of w in S_i which is closest to u_i and let w_{i+2} be the neighbor of w in S_{i+2} which is closest to v_{i+1}. (Possibly $w_i = u_i$ or $w_{i+2} = v_{i+1}$). There is a $3PC(v_{i+1}, u)$:

$P_1 = v_{i+1}, v, v_i, u$;

$P_2 = v_{i+1}$, the $v_{i+1}w_{i+2}$-subpath of S_{i+2}, w, w_i, the w_iu_i-subpath of S_i, u.

$P_3 =$ the $v_{i+1}u_{i+1}$-subpath of S_{i+1}, u. $\square$

Lemma 2.12 *Let (W, v) be a small even wheel and assume that a Type 4 node exists. Then $A(W, v)$ contains all Type 2, 3 and 4 nodes and*

$$\left| \bigcap_{u \in A(W,v)} N_W(u) \right| \geq 2.$$

Proof: Let z be a Type 4 node, having painted neighbors z_i and z_j of distinct colors in (W, v). We show that each of the two u_iu_j-subpaths of W contains an intermediate node in $\bigcap_{u \in A(W,v)} N_W(u)$.

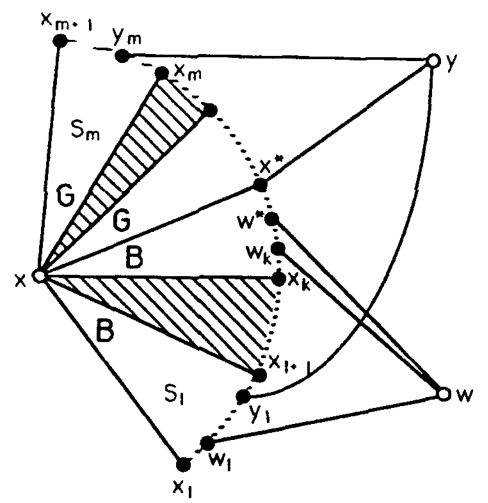

Figure 4:

Let P be one of the two u_iu_j-subpath of W and let X be the set of Type 2, 3, 4 nodes in (W,v). Pick a pair of nodes x, $y \in X$ such that y is of Type 4 relative to (W,x) and y has two painted neighbors y_l, y_m in P of distinct colors in (W,x). Furthermore nodes x and y are chosen such that the length of the y_ly_m-subpath $P_{y_ly_m}$ of P is shortest. If $P_{y_ly_m}$ contains exactly one unpainted node of (W,x), then the proof follows from Lemma 2.11. Now consider the case where $P_{y_ly_m}$ contains more than one unpainted neighbor. This number is odd, say $2k+1$, and let x^* be the $k+1^{st}$ unpainted node in $P_{y_ly_m}$, starting from either end. We show that every Type 2, 3, 4 node with respect to (W,x) is adjacent to x^*.

Assume not. Then there exists a node w of Type 3 or 4 that is not adjacent to x^*. Let S_l and S_m be the sectors of (W,x) containing y_l and y_m respectively and having x_l, x_{l+1} and x_m, x_{m+1} as endnodes, see Figure 4.

Let $P_{x_lx_{m+1}}$ be the x_lx_{m+1}-subpath of W containing x^*. Then by assumption, in $P_{x_lx_{m+1}}$, node w has either two neighbors in every green sector of (W,x) or two neighbors in every blue sector. Assume w.l.o.g. that the second alternative holds and that S_l is painted blue. Let x_k be the other endnode of the blue sector having x^* as endnode. In this sector, let w^* be the neighbor of w closest to x^* and w_k the one closest to x_k.

If w has at least one unpainted neighbor in S_l, then let w_l be such a neighbor and assume w.l.o.g. that the y_lw_l-subpath $P_{y_lw_l}$ of S_l does not

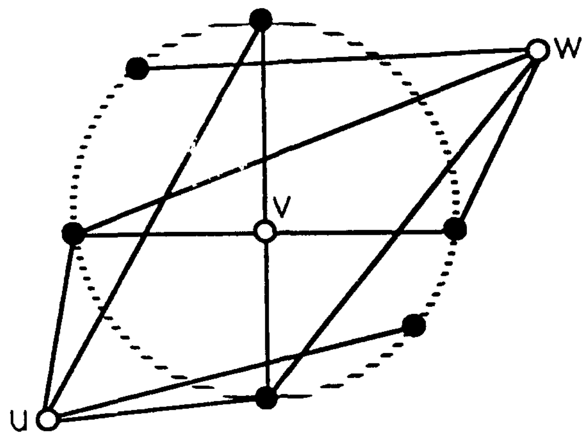

Figure 5: Weakly nested nodes u and w.

contain another neighbor of w or y. Then the following three paths induce a $3PC(x^*, w)$.

$$P_1 = x^*, y, y_l, P_{y_l w_l}, w_l, w; \quad P_2 = x^*, x, x_k, P_{x_k w_k}, w_k, w; \quad P_3 = x^*, P_{x^* w^*}, w^*, w.$$

Hence w is adjacent to x_l and x_{l+1}. In the wheel (W, w), node y is of Type 4, having neighbors x^* and y_l in sectors of (W, w) of opposite colors. But now the $x^* y_l$-subpath of P is shorter than $P_{y_l y_m}$, a contradiction to the choice of the pair x, y. $\square$

We now examine the structure of the Type 3 nodes and discuss how they relate to the other strongly adjacent nodes in V^r. We continue to assume that the even wheel (W, v) is small.

Let u and w be two nodes each having more than two neighbors in W. We say that u and w are *weakly nested* if, in every sector S of (W, v) where each of the nodes u and w has two neighbors, either the neighbors of w both belong to the path connecting the two neighbors of u, or vice versa, the neighbors of u both belong to the path connecting the two neighbors of w. See Figure 5 for an example. We say that a family of nodes has the *weak nestedness property* if every pair u, w of nodes in the family is weakly nested.

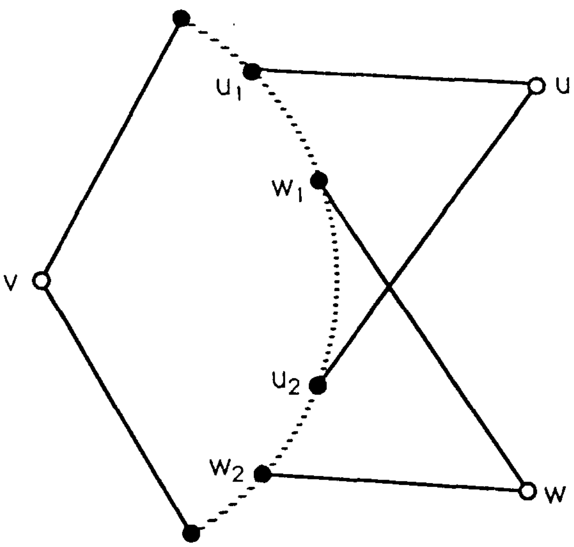

Figure 6:

Lemma 2.13 *Let (W, v) be a small even wheel. Then the weak nestedness property holds for the family comprised of the Type 2, 3 and 4 nodes.*

Proof: It follows from the definition that v is weakly nested with all other nodes. Assume that two nodes $u, w \neq v$, each having more than two neighbors in W, are not weakly nested. There are two ways in which this could happen.

Case 1: There is a sector S where the neighbors of u, say u_1 and u_2, and the neighbors of w, say w_1 and w_2, appear in the order u_1, w_1, u_2, w_2 where $u_1 \neq w_1, u_2 \neq w_1$ and $u_2 \neq w_2$, see Figure 6. In the wheel (W, u), node w has a unique neighbor, namely w_1, in one sector and w_1 is painted. This contradicts Lemma 2.4.

Case 2: There is a sector S_i where the neighbors of u, say u_1 and u_2, and the neighbors of w, say w_1 and w_2, appear in the order u_1, u_2, w_1, w_2 (possibly $u_2 = w_1$), see Figure 7. With the notation of Figure 7, let u_3 be the neighbor of u in $(V(S_{i+1}) \setminus \{v_{i+1}\}) \cup V(S_{i+2})$ which is closest to v_{i+1}, in the path defined by S_{i+1} and S_{i+2}. Similarly, let w_3 be the neighbor of w in $(V(S_{i+1}) \setminus \{v_{i+1}\}) \cup V(S_{i+2})$ closest to v_{i+1}. The nodes u_3 and w_3 exist as a consequence of Definition 2.5. Since w is not adjacent to v_{i+1}, it follows from Lemma 2.3 that $w_3 \in V(S_{i+2})$. As a consequence of Definition 2.5, node u can have at most two neighbors

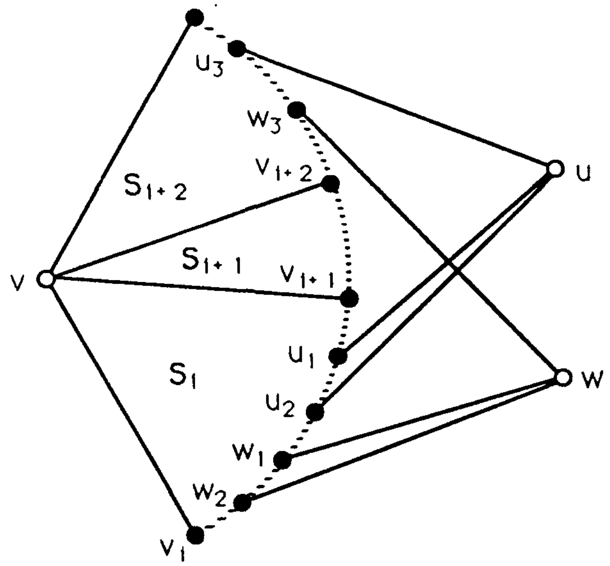

Figure 7:

in a sector of (W, w). This implies that u_3 does not belong to the w_1w_3-subpath of W containing S_{i+1}. Define the hole W' as follows. It contains the edge (u, u_2), the u_2v_i-subpath of S, the edges v_iv, vv_{i+2}, the $v_{i+2}u_3$-subpath of S_{i+2} and finally the edge u_3u. W' is shorter than W and node w has at least three neighbors in W, contradicting the assumption that (W, v) is small. □

Let $u, w \in V^r$ each have more than two neighbors in W. We say that $u \succ w$ relative to the wheel (W, v) if, in every sector S of (W, v) where each of the nodes u and w has two neighbors, the neighbors of w both belong to the path connecting the two neighbors of u in S. We say that a family of nodes has the *nestedness property* if, for every pair u, w of such nodes, either $u \succ w$ or $w \succ u$ or both.

Lemma 2.14 *Let (W, v) be a small even wheel. If neither $w \succ u$ nor $u \succ w$ holds relative to (W, v), then w is a Type 4 node relative to the wheel (W, u) and u is Type 4 relative to (W, w).*

Proof: By Lemma 2.13, nodes u and w are weakly nested. This implies that, if neither $w \succ u$ nor $u \succ w$ holds, then there exist two sectors S_i, S_j in which the nodes u, w have their neighbors as in Figure 8 where, possibly, either $u_{i_1} = w_{i_1}$ or $u_{i_2} = w_{i_2}$ but not both, and, possibly, either $u_{k_1} = w_{k_1}$

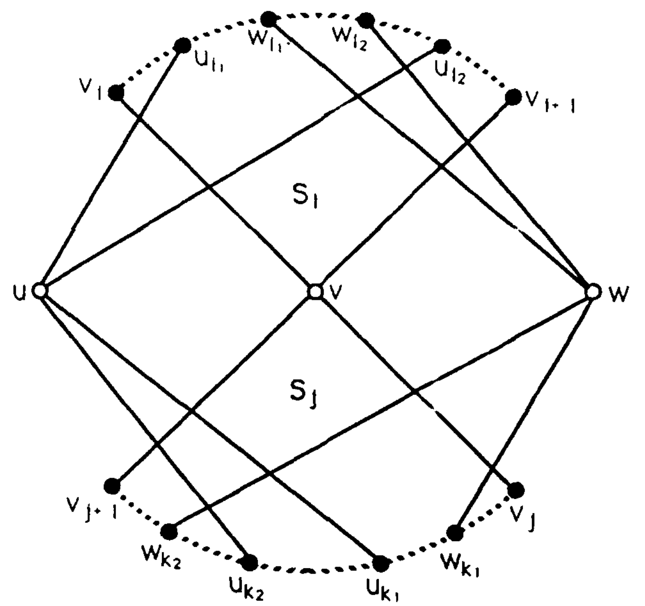

Figure 8:

or $u_{k_2} = w_{k_2}$ but not both. However, since the $u_{i_1}u_{k_2}$-subpath of W ($u_{k_1}u_{i_2}$-subpath) not containing u_{i_2} (u_{i_1}) have an even number of nodes in $N_W(v)$, it follows that, in each set $\{w_{k_1}, w_{k_2}\}, \{w_{i_1}, w_{i_2}\}$ at least one of the nodes is painted in (W, u) and the colors are distinct. The proof for u and (W, w) is identical. □

We are now ready to prove the main result of this section.

Proof of Theorem 2.2: If there exists a small even wheel having at least one Type 4 node, then the result holds as a consequence of Lemma 2.12. So, we now consider the case where every small even wheel has no Type 4 node.

Let (W, v) be a small even wheel and let B denote the set of nodes of V^r with more than two neighbors in W. Lemma 2.14 shows that the nestedness property holds for the family B. Consider the family $\mathcal{F}$ comprising the Type 2 nodes and Type 3 nodes that have painted neighbors in blue sectors. The definition of the relation $\succ$ implies that, if $x, y, z \in \mathcal{F}$ satisfy $x \succ y$ and $y \succ z$, then we also have $x \succ z$, i.e. the relation $\succ$ is transitive in the family $\mathcal{F}$. (This follows from the observations that a contradiction to $x \succ z$ must occur in a blue sector and that y also has two neighbors in that sector). Therefore, we can find $u^* \in \mathcal{F}$ such that $x \succ u^*$ for every $x \in \mathcal{F}$. Now, consider the wheel (W, u^*). Note that (W, u^*) is small and, w.l.o.g., its blue sectors are subpaths of the blue sectors of (W, v). By our choice of u^*, no node of B has a blue neighbor in (W, u^*). By Lemma 2.14, B is a nested family relative to (W, u^*). Since all Type 3 nodes relative to (W, u^*) have green neighbors, the relation $\succ$ is transitive on B. Recall that the definition of nestedness also implies that, for $x, y \in B$ either $y \succ x$ or $x \succ y$ or both. So we have a total transitive order on B.

For $i \geq 1$ integer, let $A_i(W, u^*)$ be the set of Type 2 and 3 nodes w, strongly adjacent to (W, u^*) such that $|N_W(w) \cap N_W(u^*)| \geq i$. The total transitive order of the relation $\succ$ implies that $|\bigcap_{u \in A_i(W,u^*)} N_W(u)| \geq i$. Note that the set $A(W, u^*)$ of Theorem 2.2 coincides with $A_2(W, u^*)$. Hence Theorem 2.2 holds for the wheel (W, u^*) and the proof is complete. □

Since the wheels used in the proof of Theorem 2.2 are small, we make the following observation.

Remark 2.15 *There exists a small even wheel (W, v) that satisfies Theorem 2.2.*

3 An Extended Star Cutset Theorem for Small Even Wheels

In this section, we prove the following key result concerning the decomposition of balanced bipartite graphs that contain an even wheel.

Theorem 3.1 *Let (W, v) be a small even wheel in a balanced bipartite graph. Then every path connecting a blue node to a green node of (W, v) contains a node in $N(v) \cup A(W, v)$.*

Corollary 3.2 *A balanced bipartite graph containing an even wheel has an extended star cutset.*

Proof: There exists a small even wheel (W, v) such that $\bigcap_{u \in A(W,v)} N_W(u) \neq \emptyset$, as a consequence of Theorem 2.2 and Remark 2.15. Now, for any $f_1, f_2 \in$

$\cap_{u \in A(W,v)} N_W(u)$, Theorem 3.1 implies that $N(v) \cup (N(f_1) \cap N(f_2))$ is an extended star cutset. □

A bipartite graph is *linear* if it contains no cycle of length 4. Linear balanced bipartite graphs have been studied by Conforti and Rao [11], who have proven the following star cutset theorem. This theorem can now be deduced from Theorem 3.1.

Corollary 3.3 *A linear balanced bipartite graph containing an even wheel has a star cutset.*

Proof: Let (W, v) be an even wheel in a linear balanced bipartite graph. Then we have $A(W,v) = \{v\}$. Otherwise, let u be some other node in $A(W,v)$ and let f_1 and f_2 be two neighbors of u in N_{W^v}. The nodes u, v, f_1 and f_2 induce a cycle of length 4, contradicting the linearity assumption. Now let (W, v) be a small even wheel. It follows from Theorem 3.1 that $N(v)$ is a star cutset. □

In the proof of Theorem 3.1, we make use of the following lemma, which appears in [11].

Lemma 3.4 *Let (W, v) be an even wheel in a balanced bipartite graph G and let P be a chordless path with nodes in $V(G) \setminus (V(W) \cup N(v))$ such that any $x \in V(P)$ is adjacent to at most one node $N_W(v)$ and to no painted node of W. Then at most two nodes of $N_W(v)$ have at least one neighbor in P.*

Proof: Assume the lemma is not true and let $P' = y_1, y_2, \ldots, y_n$ be a shortest subpath of P with the property that three distinct nodes of $N_W(v)$ have at least one neighbor in P'. Denote by v_1, v_2, v_3 the three nodes of $N_W(v)$ with at least one neighbor in P'. We can assume w.l.o.g. that v_1 is adjacent to y_1 and no other node of P', v_3 is adjacent to y_n and no other node of P' and that v_2 is adjacent to some intermediate nodes of P'. Let y_i and y_j be such nodes, such that the y_1y_i-subpath $P'_{y_1y_i}$ of P' and the y_jy_n-subpath $P'_{y_jy_n}$ of P' are as short as possible.

Let P_{ij} be the v_iv_j-subpath of W not containing v_k, for $i,j,k \in \{1,2,3\}$ and $i \neq j \neq k$. Let H_{12} be the hole induced by the nodes in P_{12} and in $P'_{y_1y_i}$, H_{13} the hole induced by the nodes in P_{13} and in P' and let H_{23} be the hole induced by the nodes in P_{23} and in $P'_{y_jy_n}$. Since (W,v) is an even wheel, then one of the paths P_{ij} contains an odd number of intermediate nodes in $N(v)$. Let P_{12} be this path. Then (H_{12},v) is an odd wheel. □

Proof of Theorem 3.1: If the theorem does not hold for (W,v) , let $P = s^*, s, \ldots, t, t^*$ be a shortest path connecting nodes of W with distinct colors, and containing no node of $N(v) \cup A(W,v)$. W.l.o.g. assume that $v \in V^r$, s^* is green and t^* is blue. The following possibilities can occur for nodes s and t.

(a) Node s (or t) has only one neighbor in W, namely s^* (t^* respectively),

(b) Node s (or t) belongs to $V^c \setminus N(v)$, is strongly adjacent to W, but all its neighbors are in the same sector of (W,v),

(c) Node s (or t) belongs to $V^c \setminus N(v)$ and has exactly two neighbors in (W,v), one in sector S_i and one in sector S_j, $i \neq j$, where S_i and S_j have the same color,

(d) Node s (or t) belongs to $V^r \setminus A(W,v)$ and is a Type 1 node,

(e) Node s (or t) belongs to $V^r \setminus A(W,v)$ and is a Type 3 node with at most one neighbor in $N_W(v)$.

It follows from Theorem 2.1 and Lemma 2.3 that no other possibility can occur for the node s (or t).

Next, we show that we can dispose of the possibilities (b) and (d) by modifying the wheel (W,v) and the path P.

Claim 1: *There exists a wheel (W',v) and a path $P' = s^{*'}, s', \ldots, t', t^{*'}$ connecting nodes of distinct colors in (W',v), containing no node of $N(v) \cup A(W',v)$, such that the nodes s' and t' satisfy one of the properties (a), (c) or (e) above and, furthermore, the nodes of $V(P') \setminus \{s^{*'}, s', t', t^{*'}\}$ have at most one neighbor in W'.*

Proof: First, assume that some node u of $V(P)\setminus\{s^*, s, t, t^*\}$ has at least two neighbors in W. These neighbors are unpainted, otherwise a shorter path P would exist. All Type 2 nodes are in $A(W,v)$, so u must be of Type 1. Denote by v_i and v_{i-1} the nodes of W adjacent to u and by S_i the v_iv_{i-1}-sector of (W,v)Assume w.l.o.g. that S_i is a blue sector. Construct W' from W by replacing the sector S_i by the sector v_{i-1}, u, v_i and let P' be the s^*u-subpath of P. Note that $A(W',v) = A(W,v)$. Therefore, P' connects sectors of distinct colors in (W',v) and contains no node of $N(v)\cup A(W',v)$. In P', the node t' adjacent to u is different from s (if $s = t'$, then this node is Type 4 relative to (W',v) but all Type 4 nodes belong to $A(W',v)$). Note also that P' is shorter than P. So by repeating the above procedure, we can dispose of all the nodes of $V(P)\setminus\{s^*, s, t, t^*\}$ with at least two neighbors in W. In the remainder, we assume w.l.o.g. that the nodes of $V(P)\setminus\{s^*, s, t, t^*\}$ have at most one neighbor in W and, if this neighbor exists, it is unpainted.

Assume that s satisfies property (b) or (d) and let S_i be the sector containing s^*. Denote by v_i and v_{i-1} the endnodes of S_i and by s_i and s_{i-1} the neighbors of s in S_i that are closest to v_i and v_{i-1} respectively. Let (W',v) be the wheel obtained from (W,v) by substituting the $s_{i-1}s_i$-subpath of S_i with s_{i-1}, s, s_i and let P' be the subpath obtained from P by removing the node s^*, namely $P' = s, s', \ldots, t, t^*$. Since $A(W',v) = A(W,v)$, the path P' connects two sectors of (W',v) with distinct colors and contains no node of $N(v)\cup A(W',v)$. Note that $s' = t$ cannot occur, since this node would be Type 4 relative to (W',v), a contradiction to the fact that P' contains no node of $A(W',v)$. Therefore, s' has at most two neighbors in W'. If s' does have two neighbors, it must be of Type 1 relative to (W',v), i.e. Property (d) holds. In this case the above procedure can be repeated and P' can be shortened again. The proof of Claim 1 is now complete

As a consequence of this claim, we can assume w.l.o.g. that (W,v) and $P = s^*, s, \ldots, t, t^*$ have the following properties, in addition to those already stated at the beginning of the proof: s and t satisfy Properties (a), (c) or (e) and the nodes of $V(P)\setminus\{s^*, s, t, t^*\}$ have at most one neighbor in W.

Claim 2: *Let s be a Type 3 node.*

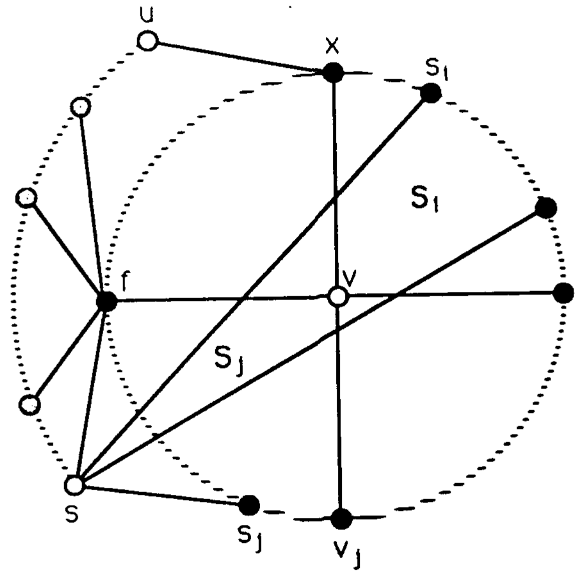

Figure 9:

(i) If $N_W(s) \cap N_W(v) = \emptyset$, then no node of $V(P) \setminus \{s^*, s, t, t^*\}$ is adjacent to a node of W.

(ii) If $N_W(s) \cap N_W(v) = \{f\}$, then no node of $V(P) \setminus \{s^*, s, t, t^*\}$ is adjacent to a node of $W \setminus \{f\}$.

Proof: Assume not and let u be the node of $V(P) \setminus \{s^*, s, t, t^*\}$ which is closest to s in P and adjacent to a node of W (Case (i)) or of $W \setminus \{f\}$ (Case (ii)). By Claim 1, node u can only be adjacent to one node of W and this node is unpainted. Let $x \in N_W(v)$ be this node, see Figure 9.

By Remark 2.9, node s has exactly two neighbors in each green sector of (W, v). By Property (e), s has at most one unpainted neighbor in W. Let S_i be the green sector having x as endnode and let s_i be the neighbor of s closest to x in S_i. Let S_j be a green sector distinct from S_i, say with endnodes v_j and v_{j-1} and let s_j and s_{j-1} be the neighbors of s in S_j, closest to v_j and v_{j-1} respectively. Assume w.l.o.g. that s_j is painted. Then $x \in V^c$ and $s \in V^r$ are connected by a $3PC(x, s)$:

$P_1 = x, u$, the us-subpath of P, s

$P_2 = x$, the xs_i-subpath of S_i , s

$P_3 = x, v, v_j$, the $v_j s_j$-subpath of S_j, s.

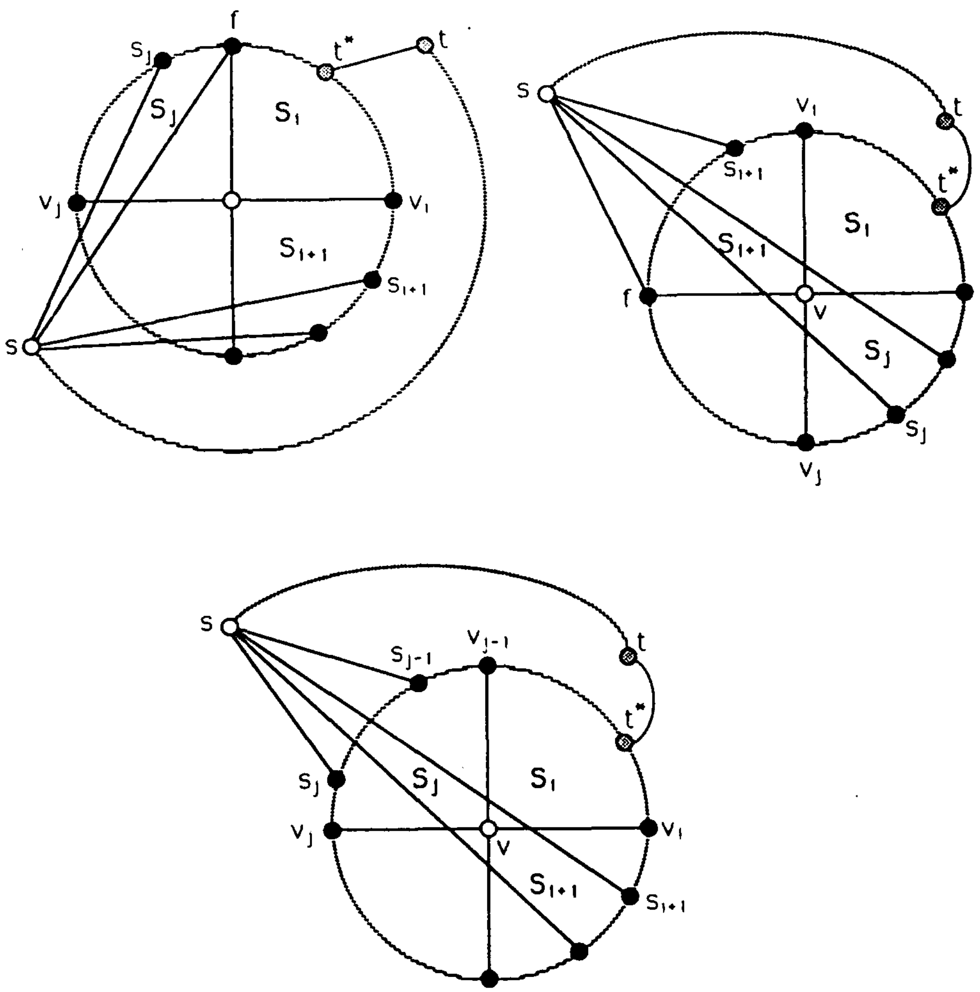

Figure 10:

This completes the proof of Claim 2.

A similar statement to Claim 2 holds when t is of Type 3.

Claim 3: *Neither node s nor node t is of Type 3.*

Proof: Assume s is a Type 3 node. By Remark 2.9, node s has exactly two neighbors in each green sector of (W, v). Furthermore, s has at most one unpainted neighbor, say f. Let S_i be the blue sector containing t^*. We choose an adjacent green sector S_{i+1} as follows, see Figure 10.

(i) If S_i has node f as an endnode, let S_{i+1} be the green sector, adjacent to S_i, which does not contain f as endnode.

(ii) If S_i does not have node f as an endnode but one of the adjacent green sectors has f as endnode, let S_{i+1} be the green sector having f as endnode.

(iii) If neither of the sectors adjacent to S_i has f as an endnode (or if f does not exist), choose S_{i+1} arbitrarily to be one of the sectors adjacent to S_i.

Let v_i be the common endnode to S_i and S_{i+1} and let s_{i+1} be the neighbor of s in S_{i+1} which is closest to v_i. Note that s_{i+1} is a painted node since $v_i \neq f$. Finally, let S_j be the green sector, distinct from S_{i+1} and adjacent to S_i. Let $v_j \neq f$ be an endnode of S_j which is not adjacent to t^* and let v_{j-1} be the other endnode of S_j. (Note that such a choice of v_j is always possible due to the above conditions (i)–(iii)). Let s_j be the neighbor of s in S_j which is closest to v_j.

Case 1: Node t is not adjacent to v_j.

Now there is a $3PC(v_i, s)$:

$P_1 = v_i$, the $v_i t^*$-subpath of S_i, the $t^* s$-subpath of P, s.

$P_2 = v_i$, the $v_i s_{i+1}$-subpath of S_{i+1}, s.

$P_3 = v_i, v, v_j$, the $v_j s_j$-subpath of S_j, s.

Case 2: Node t is adjacent to v_j.

This implies that t is a Type 3 node and that no node of $V(P) \setminus \{s^*, s, t, t^*\}$ has a neighbor in $N_W(v)$, else s or t contradicts Claim 2. Let P^* be the st-subpath of P. Then there is a $3PC(v_j, s)$.

$P_1 = v_j, t, P^*, s$.

$P_2 = v_j$, the $v_j s_j$-subpath of S_j, s.

In Case (i) or (ii), P_3 is chosen as follows:

$P_3 = v_j, v, f, s$.

In Case (iii), let s_{j-1} be the neighbor of s closest to v_{j-1} in S_j. Then P_3 is chosen as follows:

$P_3 = v_j, v, v_{j-1}$, the $v_{j-1}s_{j-1}$-subpath of S_j, s.

This completes the proof of Claim 3.

Claim 4: *If s satisfies Property (c), then there exists a green sector S with the property that each endnode of S is adjacent to at least one node in the set $V(P) \setminus \{s^*, s, t, t^*\}$.*

Proof: Since node s satisfies Property (c), Theorem 2.1 implies that there exists a S_i, say with endnodes v_i and v_{i-1}, such that the unique neighbor s_i of s in S_i is not adjacent to v_i, v_{i-1}. Let S_j be the sector containing the second neighbor s_j of s and let v_j, v_{j-1} be the endnodes of S_j. By Lemma 3.4, at most two nodes of $N_W(v)$ are adjacent to $V(P) \setminus \{s^*, s, t, t^*\}$.

Case 1: No node of $N_W(v)$ is adjacent to $V(P) \setminus \{s^*, s, t, t^*\}$.

Let S_k be the sector containing t^* and let $v_k \neq v_j, v_{j-1}$ be an endnode of S_k. We can assume w.l.o.g. that $v_i \neq v_k$ and that v_j is not adjacent to t^*. Then, there is a $3PC(v, s)$:

$P_1 = v, v_k$, the $v_k t^*$-subpath of S_k, P.

$P_2 = v, v_i$, the $v_i s_i$-subpath of S_i, s.

$P_3 = v, v_j$, the $v_j s_j$-subpath of S_j, s.

Case 2: Exactly one node of $N_W(v)$, say v_l is adjacent to $V(P) \setminus \{s^*, s, t, t^*\}$.

Starting from s, let u^* be the first neighbor of v_l encountered on P.

Case 2.1: $v_l \neq v_j, v_{j-1}$.

Assume w.l.o.g. that $v_l \neq v_i$. Then there is a $3PC(v, s)$:

$P_1 = v, v_l, u^*$, the u^*s-subpath of P, s,

P_2 and P_3 are the same as in Case 1.

Case 2.2: $v_l = v_{j-1}$ and t satisfies Property (a).

Then, the 3-path configuration of Case 1 is still valid, except if t^* is adjacent to v_j. So, we consider the case where t^* is adjacent to v_j. Let

Q be the $v_{j-1}t^*$-subpath of W not containing v_j. Let v^* be the neighbor of v_{j-1} which is closest to t^* on P and let P^* be the t^*v^*-subpath of P. Let H be the hole formed by Q, P^* and the edge $v_{j-1}v^*$). Then (H, v) is an odd wheel.

Case 2.3: $v_l = v_{j-1}$ and t satisfies Property (c).

As a consequence of Theorem 2.1, one of the neighbors of t in W is adjacent to no node of $N_W(v)$. Choose t^* to be such a neighbor of t. Then the argument of Case 1 still holds.

Case 3: Two nodes of $N_W(v)$ are adjacent to $V(P) \setminus \{s^*, s, t, t^*\}$.

Starting from s, let u^* be the first node of P having a neighbor in $N_W(v)$, say $v_l \in N_W(v)$. If $v_l \neq v_{j-1}$, then the argument of Case 2.1 still holds. So, assume w.l.o.g. that $v_l = v_{j-1}$. Let v_p be the other node of $N_W(v)$ with neighbors in $V(P) \setminus \{s^*, s, t, t^*\}$. Starting from s, let w^* be the first neighbor of v_p encountered on P. Assume w.l.o.g. that $v_i \neq v_p$. If $v_p \neq v_j$, then there is $3PC(v, s)$:

$P_1 = v, v_p, w^*$, the w^*s-subpath of P, s.

P_2 and P_3 are as in Case 1.

Hence v_p and v_l are the endnodes of the green sector S_j and the claim follows.

If both s and t satisfy Property (c), then Claim 4 implies that at least three nodes of $N_W(v)$, namely the endnodes of two sectors of distinct colors, have neighbors in $V(P) \setminus \{s^*, s, t, t^*\}$. This contradicts Lemma 3.4 asserting that at most two nodes of $N_W(v)$ can have neighbors in $V(P) \setminus \{s^*, s, t, t^*\}$. So we can assume w.l.o.g. that t satisfies Property (a). The next claim shows that this cannot occur either, proving the theorem.

Claim 5: *Node t cannot satisfy Property (a).*

Proof: Assume t satisfies Property (a) and let v_j, v_{j-1} be the endnodes of the sector S_j containing t^*. First, we show that at least one node of $N_W(v)$ has a neighbor in $V(P) \setminus \{s^*, s, t, t^*\}$. Assume not. Then Claim 4 implies that node s satisfies Property (a). Let P_1 and P_2 be the two s^*t^*-subpaths of W. Let H_1 (H_2) be the hole formed by P and P_1 (P_2 respectively). Both H_1 and H_2 have an odd number of neighbors of v and, for at least one of the holes this number is greater than one. So either (H_1, v) or (H_2, v) is an odd wheel.

When traversing P from t^*, let u^* be the first node encountered which has a neighbor v_l in $N_W(v)$. We show that $v_l = v_j$ or v_{j-1}. Assume not and let P_1 and P_2 be the t^*v_j-subpaths of W and let P^* be the t^*u^*-subpath of P. Let H_1, H_2 be the two holes formed by P^*, the edge u^*v_j and P_1, P_2 respectively. Note that v has an odd number of neighbors in one of these two holes. This odd number is greater than one, since $v_l \neq v_j, v_{j-1}$. Hence, either (H_1, v) or (H_2, v) is an odd wheel. So v_l is an endnode of S_j. Assume w.l.o.g. that $v_l = v_j$.

If v_j is the only node of $N_W(v)$ with neighbors in $V(P) \setminus \{s^*, s, t, t^*\}$ then, by Claim 4, node s satisfies Property (a) and, therefore, v_j must also be an endnode of the sector containing s^*. In other words, s^* and t^* belong to adjacent sectors. Let H be the hole formed by P and the s^*t^*-subpath of W which does not contain v_j. Then (H, v) is an odd wheel.

So there must be a second node of W, say $v_i \neq v_j$, with neighbors in $V(P) \setminus \{s^*, s, t, t^*\}$. Let w be the neighbor of v_i in P which is the closest to t^* and let Q be the wt^*-subpath of P. Let w^* be the neighbor of v_j closest to w in Q, and let Q^* be the ww^*-subpath of Q. Let P_1 be the v_iv_j-subpath of W which does not contain t^*. Let P_2 be the v_it^*-subpath of W which does not contain v_j. Finally, define the holes H_1 and H_2 as follows. H_1 is formed by P_1, Q^* and the edges wv_i, w^*v_j. H_2 is formed by P_2, Q and the edge wv_i. One of the holes H_1, H_2 contains an odd number of neighbors of v and, if v_i is not an endnode of the sector containing t^*, the number of neighbors of v is greater than one in each of the holes H_1 and H_2. So, either (H_1, v) or (H_2, v) is an odd wheel. So $v_i = v_{j-1}$.

If s satisfies Property (a), then the same argument shows that v_{j-1} and v_j are also the endnodes of the sector containing s^*, a contradiction to the fact that s^* and t^* are in different sectors. If s satisfies Property (c), then, by Claim 4, v_{j-1} and v_j are the endnodes of a green sector, a contradiction to the fact that the sector containing t^* is painted blue. Hence the claim follows and the proof of Theorem 3.1 is now complete. □

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In this seven part paper, we prove the following theorem:

At least one of the following alternatives occurs for a bipartite graph G:

- The graph G has no cycle of length $4k+2$.
- The graph G has a chordless cycle of length $4k+2$.

- There exist two complete bipartite graphs K_1, K_2 in G having disjoint node sets, with the property that the removal of the edges in K_1, K_2 disconnects G.
- There exists a subset S of the nodes of G with the property that the removal of S disconnects G, where S can be partitioned into three disjoint sets T, A, N such that $T \neq \emptyset$, some node $x \in T$ is adjacent to every node in $A \cup N$ and, if $|T| \geq 2$, then $|A| \geq 2$ and every node of T is adjacent to every node of A.

A 0,1 matrix is balanced if it does not contain a square submatrix of odd order with two ones per row and per column. Balanced matrices are important in integer programming and combinatorial optimization since the associated set packing and set covering polytopes have integral vertices.

To a 0,1 matrix A we associate a bipartite graph $G(V^r, V^c; E)$ as follows: The node nets V^r and V^c represent the row set and the column set of A and edge ij belongs to E if and only if $a_{ij}=1$. Since a 0,1 matrix is balanced if and only if the associated bipartite graph does not contain a chordless cycle of length $4k+2$, the above theorem provides a decomposition of balanced matrices into elementary matrices whose associated bipartite graphs have no cycle of length $4k+2$. In Part VII of the paper, we show how to use this decomposition theorem to test in polynomial time whether a 0,1 matrix is balanced.